Senior Mathematical Challenge

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Solutions

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	A	B	C	B	D	A	D	E	D	D	A	C	B	C	B	E	C	C	D	B	A	E	E	A

1. What is two-fifths of the recurring decimal $0.\dot{2}\dot{5}$?

A $0.\dot{1}$ B $0.0\dot{1}$ C $0.\dot{0}\dot{1}$ D 0.10 E $0.\dot{1}\dot{0}$

Solution **E**

Written as a fraction, $0.\dot{2}\dot{5}$ is $\frac{25}{99}$. One-fifth of the number is $\frac{5}{99}$ so two-fifths $= \frac{10}{99} = 0.\dot{1}\dot{0}$.

2. A ***twip*** is a very short unit of length, derived from imperial units, and is equal to approximately 0.000018 metres. A ***league*** is a long unit of length which is equal to approximately 4800 metres.

Roughly how many twips are there in a league?

A 270 000 000 B 27 000 000 C 2 700 000 D 270 000 E 27 000

Solution **A**

The number of ***twips*** in a ***league*** is $\dfrac{4800\,\text{m}}{0.000018\,\text{m}} = \dfrac{4.8 \times 10^3}{1.8 \times 10^{-5}} = \dfrac{8}{3} \times 10^8 = 2.\dot{6} \times 10^8 \approx$ 270000000.

3. Two standard dice are placed on a table, with one on top of the other, so that only nine of the faces of the dice may be seen. The touching faces have the same number on them. The sum of the numbers on the visible faces is 33.

What is the number on the touching faces?

A 1 B 2 C 3 D 4 E 6

Solution **B**

The numbers on opposite faces of a standard dice sum to 7. On the bottom dice, there are two pairs of opposite faces which are visible. On the top dice, there are again two pairs of opposite faces which are visible along with the number on the top of that dice, n say. Therefore, $(2 + 2) \times 7 + n = 33$ and so $n = 5$. The number on the touching faces is then $7 - 5 = 2$.

4. The sizes of the three angles in a triangle, in degrees, are x, $7x$ and x^2.

What is the size of the largest angle?

A 10° B 18° C 100° D 120° E 121°

Solution **C**

As the angle sum of a triangle is 180°, $x + 7x + x^2 = 180$. Therefore, $x^2 + 8x - 180 = 0$ which factorises to $(x + 18)(x - 10) = 0$. As $x > 0$, $x = 10$. The angles are then 10°, 70° and 100°, so the largest angle is 100°.

5. When $4^5 \times 5^4$ is correctly calculated, how many digits are there in the answer?

A 4 B 6 C 10 D 16 E 20

Solution **B**

$4^5 \times 5^4 = 2^{10} \times 5^4 = 2^6 \times 2^4 \times 5^4 = 2^6 \times 10^4 = 640000$. Hence the answer has six digits.

6. One face of a solid polyhedron is an octagon.

What is the smallest possible number of edges the solid could have?

A 9 B 10 C 12 D 16 E 24

Solution **D**

The solid with the minimum number of edges must contain an octagonal 'base'. This has 8 edges and 8 vertices. Including one extra edge from each of these vertices, all of which meet at a single vertex that is not on the base, creates an octagonal pyramid. There are $8 + 8 = 16$ edges.

7. Which is the largest prime factor of $3^8 - 1$?

A 41 B 37 C 31 D 29 E 23

Solution **A**

In order to write the expression $3^8 - 1$ as the product of its primes without first calculating its value, we can use the difference of two squares: $3^8 - 1 = (3^4+1)(3^4-1) = 82\times80 = 41\times2\times2^4\times5$. Hence the largest prime factor is 41.

8. In the following expressions, x is non-zero. When one of these expressions is removed, the mean of the remaining four is $11x$.

Which expression is removed?

A $4x$ B $8x$ C $12x$ D $16x$ E $20x$

Solution **D**

The mean of four terms is $11x$ so the sum of those four terms is $44x$. As the total of the five options is $4x + 8x + 12x + 16x + 20x = 60x$, the term to exclude from the sum is $60x - 44x = 16x$.

9. A palindromic number is one where the digits read the same forwards as backwards, such as 123 321.

What is the hundreds digit of the largest six-digit palindromic number that is divisible by 18?

A 9 B 7 C 5 D 3 E 1

SOLUTION **E**

For the number we seek to be divisible by 18, it must be divisible by both 2 and 9. To be even, it must have a last digit of 0, 2, 4, 6 or 8. This last digit is also the first digit of the palindromic number, so we choose it to be an 8. To be divisible by 9, the sum of all six digits must be a multiple of 9. As we already have two 8s, the remaining four middle digits, with a maximum sum of $4 \times 9 = 36$, could sum to 2, 11, 20 or 29. The sum of these four digits must itself be even as there are two repeated pairs. So we require the second and third numbers from the left to sum to $\frac{20}{2} = 10$ at the same time as maximising the second digit due to its place value. Hence the number we seek is 891198 and the hundreds digit is a 1.

10. The prime factorization of 2024 is $2^3 \times 11 \times 23$.

How many two-digit numbers are factors of 2024?

A 2 B 4 C 6 D 7 E 8

SOLUTION **D**

As $2024 = 2 \times 2 \times 2 \times 11 \times 23$, the two-digit numbers which are factors of 2024 are either suitable multiples of 11: 11, $11 \times 2 = 22$, $11 \times 2 \times 2 = 44$, $11 \times 2 \times 2 \times 2 = 88$ or multiples of 23: 23, $23 \times 2 = 46$, $23 \times 2 \times 2 = 92$. In total, this gives seven two-digit factors.

11. Which one of the following expressions is a square number for each positive integer n?

A $n + 1$
B $n(n + 1) + 1$
C $n(n + 1)(n + 2) + 1$
D $n(n + 1)(n + 2)(n + 3) + 1$
E $n(n + 1)(n + 2)(n + 3)(n + 4) + 1$

SOLUTION **D**

Choosing n to be 1, say, eliminates options A, B and C as the resulting values 2, 3 and 7 are not square. Similarly, choosing n to be 2 eliminates option E as 721 is not square. Now considering option D, we can see that $n(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1$ which can be written as $(n^2 + 3n + 1)^2$ and is therefore a square for each positive integer n.

12. p, q, r and s are two-digit primes which between them use all the non-zero digits except 5.

What is the value of $p + q + r + s$?

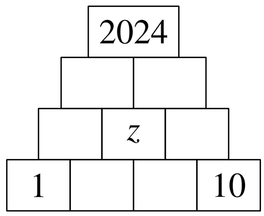
A 220 B 210 C 200 D 190 E more information needed

SOLUTION **A**

In order to create two-digit primes, the units digits must be 1, 3, 7 and 9 in some order. Therefore the tens digits must be 2, 4, 6 and 8 in some order. We can add together all the tens and all the units without being concerned which combines with which to create the actual primes, so $p + q + r + s = (20 + 40 + 60 + 80) + (1 + 3 + 7 + 9) = 220$. There are in fact four possible ways to assign the digits to create p, q, r and s.

13. The diagram shows a partially completed number pyramid. When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.

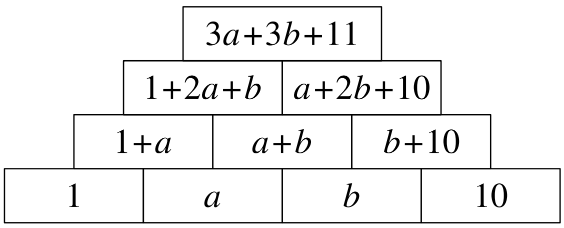
What number should appear on the brick marked 'z'?



A 176 B 617 C 671 D 716 E 761

SOLUTION **C**

Labelling the missing numbers in the bottom row as a and b, the pyramid can be filled as shown. Therefore $3a+3b+11 = 2024$, and so subtracting 11 and dividing by 3 gives $a + b = 671$. Hence 671 appears on the brick marked z.



14. P, Q, R, S and T are the digits 1, 2, 3, 4 and 5 in some order. 'PRT' and 'QRS' are both three-digit primes.

Which digit is R?

A 1 B 2 C 3 D 4 E 5

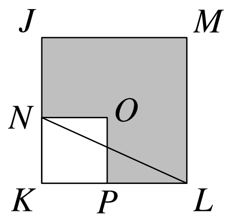
SOLUTION **B**

In order to be primes, the three-digit numbers cannot end in 2, 4 or 5. S and T must therefore be 1 and 3 in either order. We may assume that $S = 3$ so that $T = 1$. Now Q and R cannot be 2 and 4 in either order nor 4 and 5 in either order as then QRS would be a multiple of 3. Therefore Q and R must be 2 and 5 in some order. However $253 = 23 \times 11$. So $QRS = 523$ which is (and in the context of the SMC, must be) a prime. Therefore $R = 2$ and then $PRT = 421$ which is also a prime.

15. The diagram shows two squares, $JKLM$ and $NKPO$.
The length of NL is 10 cm. The shaded region has area 62 cm^2.

What is the length of KN in cm?

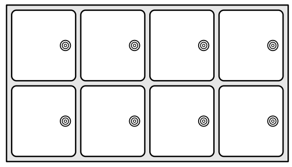
A 3 B $\sqrt{18}$ C $\sqrt{19}$ D $\sqrt{22}$ E 5



SOLUTION **C**

Let $KL = x$ cm and $KN = y$ cm. As the shaded area is 62, $x^2 - y^2 = 62$. Applying Pythagoras' Theorem to triangle NKL gives $x^2 + y^2 = 10^2$. Subtracting the first equation from the second gives $2y^2 = 38$ so $y^2 = 19$ and hence $y = \sqrt{19}$.

16. A set of cupboards containing eight identical blue doors is arranged in a 2 by 4 grid as shown. A fussy decorator wishes to paint three of the doors red such that at least one door in each row is painted red and at least two of the four corners are painted red.

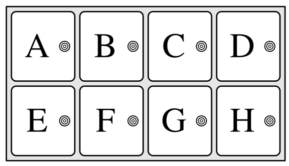


How many ways are there to do this?

A 12 B 24 C 36 D 40 E 56

SOLUTION **B**

Let the doors be labelled A to H as shown. If all three red doors are corners, the one which remains blue can be A, D, E or H. This gives 4 ways. Now suppose that just two corners are red. If those two corners are in different rows, (that is AE, AH, DE or DH) then the third red door is any one of B, C, F or G giving four ways for each of the four cases. This gives another 16 ways.



Finally, suppose the two red corner doors are in the same row. If they are AD then the third door must be F or G and if they are EH the third must be B or C. This gives a further 4 ways. So the total number of ways is $4 + 16 + 4 = 24$.

17. A bag contains four balls each of which is coloured either red or white. If one ball is drawn at random from the bag but not replaced and then a second ball is drawn at random, the probability that both balls are red is $\frac{1}{2}$.

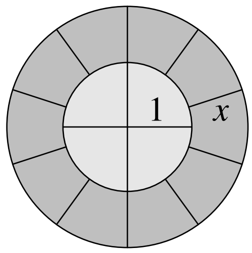
What is the probability that both balls are white?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{6}$ E 0

SOLUTION **E**

We start by finding how many red balls are in the bag. Let this number be n. Therefore the probability of two reds is $\frac{n}{4} \times \frac{(n-1)}{3} = \frac{1}{2}$. This rearranges to give $n^2 - n - 6 = 0$, which factorises to $(n-3)(n+2) = 0$. As $n \geq 0$, $n = 3$. So there are three red balls and only one white. Hence the probability that both balls are white is 0.

18. The diagram shows two concentric circles divided by radial lines into 14 pieces of equal area. The radius of the smaller circle is 1.



What is the length, x, of an outer radial line?

A $\sqrt{14} - 1$ B $\sqrt{14} - 2$ C $\dfrac{\sqrt{14}}{2} - 1$ D $\dfrac{\sqrt{14}}{2} - 2$ E $\dfrac{\sqrt{14} - 1}{2}$

SOLUTION **C**

The area of each of the four central quadrants is $\frac{1}{4} \times \pi \times 1^2 = \frac{\pi}{4}$. Therefore the area enclosed by the outer circle is $\frac{14\pi}{4}$. The radius of the outer circle is $1 + x$, therefore $\pi \times (1 + x)^2 = \frac{14\pi}{4}$, which rearranges to $2x^2 + 4x - 5 = 0$. As $x > 0$, the quadratic formula leads us to $x = \frac{\sqrt{14}}{2} - 1$.

19. Five friends are dealt two cards each from a set of twelve cards. The cards are numbered 1 to 12 inclusive. In turn, the friends declare the sum of the values of their two cards. Paolo scores 4, Quinn scores 11, Romy scores 16, Stephen scores 19 and Thomas scores 20.

Which of the following statements is true?

A Paolo has card 2　　B Quinn has card 3　　C Romy has card 5
D Stephen has card 7　　E Thomas has card 11

Solution **C**

In order to find the correct option, we will try to determine which cards are held by which friend. Here is a list of pairs of cards that are feasible for each friend, given their declared totals.

Pablo (4)	Quinn (11)	Romy (16)	Stephen (19)	Thomas (20)
3, 1	10, 1	12, 4	12, 7	12, 8
	9, 2	11, 5	11, 8	11, 9
	8, 3	10, 6	10, 9	
	7, 4	9, 7		
	6, 5			

The total of all the cards 1 to 12 is 78. The cards held by the friends sum to $4+11+16+19+20 = 70$ so the unused cards sum to 8. Paolo's total is 4 so he has 1 and 3. The unused cards must then be 2 and 6.

The only possibilities for Thomas to have 20 are 12, 8 or 11, 9. Suppose that Thomas has 11, 9. Then Stephen must have 12, 7. However, then there is no way for Romy to have 16. Hence Thomas must have 12, 8. Then Stephen has 10, 9, Romy has 11, 5 and Quinn has 7, 4.

Pablo (4)	Quinn (11)	Romy (16)	Stephen (19)	Thomas (20)
3, 1	7, 4	11, 5	10, 9	12, 8

Of the options given, only *C* is true.

20. Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$. What is the maximum possible value of y?

A 40　　B 60　　C 240　　D 420　　E 480

Solution **D**

Rearranging $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$ gives $y = \frac{20x}{x-20} = 20 + \frac{400}{x-20}$. In order to maximise y, we require $x - 20$ to be as small as possible. As x is an integer, $x = 21$. Then $y = 20 + \frac{400}{21-20} = 420$.

21.

Across
1. A multiple of 9
3. A square

Down
1. A multiple of 11
2. A multiple of 13 and of 19

1		2
	■	
3		

The crossnumber is to be filled with eight of the digits 1 to 9, which are each used once.

Which digit is not used?

A 9 B 8 C 5 D 3 E 2

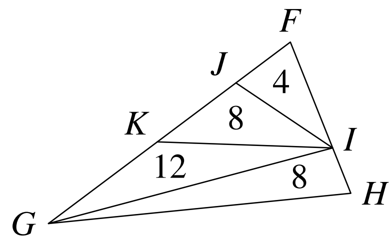
SOLUTION **B**

We begin where we have least choice. Here that is '2 Down'. The smallest multiple of both 13 and 19 is $13 \times 19 = 247$. The list of all such three-digit multiples is 247, 494, 741 and 988. As digits may not be repeated we have only 247 and 741. The units digit of '2 Down' is also the units digit of '3 Across', 'A square'. As squares do not end in 7, '2 Down' must be 741. Considering '3 Across', three-digit squares which end in 1 come from $11^2, 21^2, 31^2, 19^2$ or 29^2. However, without repeated digits or use of 4 or 7, our only possibilities are $31^2 = 961$ or $19^2 = 361$. Now considering '1 Down' we look for multiples of 11 which end in either 3 or 9. Multiples ending in 3 come from the answers to $11 \times 13, 11 \times 23, \ldots, 11 \times 83$. Given the remaining available digits, this is either $11 \times 23 = 253$ or $11 \times 53 = 583$. Multiples ending in 9 come from the answers to $11 \times 19, 11 \times 29, \ldots, 11 \times 89$. Given the remaining available digits, this can only be $11 \times 49 = 539$. At this stage we have three cases under consideration.

To complete the crossnumber with a multiple of 9 in '1 Across', we require the digits in the top row to sum to a multiple of 9. If '1 Down' were to be either 539 or 583 the middle digit would need to be a 6 so that $5 + 6 + 7 = 18$. However the 6 has already been used. '1 Down' must therefore be the only remaining possibility, 253, and '1 Across' must be 297. The completed grid is as shown. The digit which is not used is 8.

[1] 2	9	[2] 7
5	■	4
[3] 3	6	1

22. As shown in the diagram, triangle FGH is divided into four smaller triangles which have areas 4, 8, 12 and 8 respectively.



What is the area of triangle IKH?

A 4 B 5 C 6 D 7 E 8

Solution **A**

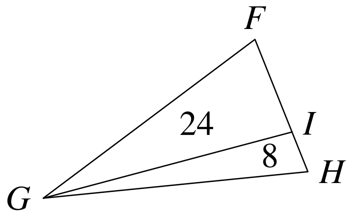


Fig. 1

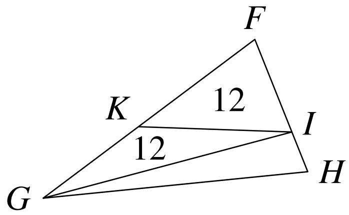


Fig. 2

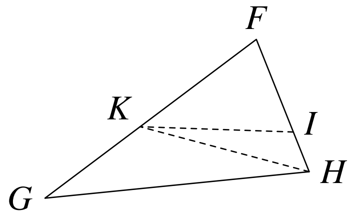


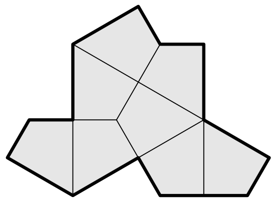
Fig. 3

Using 'area of a triangle = $\frac{1}{2}$ base × perpendicular height' with FH as the base, triangles FGI and FGH have the same perpendicular height. Their areas are therefore in the same proportions as the lengths of their bases and so $IH = \frac{1}{4}FH$.

Now viewing FI as the base of both triangles FGI and FKI, we can deduce that the perpendicular height from FI to K is half the perpendicular height from FI to G.

The area of triangle $IKH = \frac{1}{2}\ IH \times$ the perpendicular height from IH to $K = \frac{1}{2} \times \frac{1}{4}FH \times \frac{1}{2}$ the perpendicular distance from IH to $G = \frac{1}{8} \times$ the area of triangle $FGH = \frac{1}{8} \times 32 = 4$.

23. The plane can be tiled using the 'hat tile' shown here. This tile can be subdivided into eight congruent kites. The area of the hat tile is $8\sqrt{3}$.

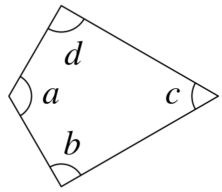


What is the perimeter of the hat tile?

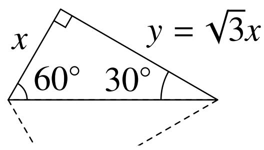
A $8 + 12\sqrt{3}$ B $16 + 6\sqrt{3}$ C $8 + 8\sqrt{3}$
D $6 + 8\sqrt{3}$ E $8 + 6\sqrt{3}$

SOLUTION **E**

On the diagram shown in the question, we can see that each kite has a line of symmetry. Therefore $\angle b = \angle d$. Also, where three kites meet at a point with no gaps, $\angle a = 120°$. Where four kites meet at a point with no gaps, $\angle b = \angle d = 90°$. As the angle sum of a quadrilateral is $360°$, $\angle c = 60°$. Each half-kite is therefore a $30°, 60°, 90°$ triangle, with lengths in the ratio $1 : \sqrt{3} : 2$.



Let the perpendicular lengths be x and y as shown. So $y = \sqrt{3}x$. As the area of the whole hat tile is $8\sqrt{3}$, the area of each kite is $\sqrt{3}$. Therefore $2 \times \frac{xy}{2} = \sqrt{3}$, so $\sqrt{3}x^2 = \sqrt{3}$ and $x = 1$. Therefore $y = \sqrt{3}$. The perimeter of the hat tile is $8x + 6y = 8 \times 1 + 6 \times \sqrt{3} = 8 + 6\sqrt{3}$.



24. A function f satisfies the equation $f(x) + f\left(\frac{1}{1-x}\right) = 24x$ for all real values of x except $x = 0$ and $x = 1$.

What is the value of $f(3)$?

A 40 B 42 C 45 D 48 E 50

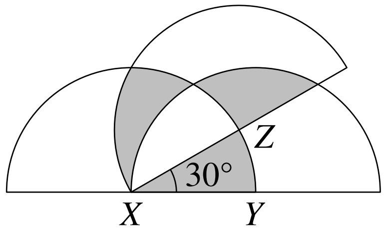
SOLUTION **E**

First let $x = 3$, then $f(3) + f(\frac{1}{1-3}) = 24 \times 3$. Therefore $f(3) + f(-\frac{1}{2}) = 72$ (a). Now let $x = -\frac{1}{2}$, then $f(-\frac{1}{2}) + f(\frac{1}{1-\frac{-1}{2}}) = 24 \times -\frac{1}{2}$. So $f(-\frac{1}{2}) + f(\frac{2}{3}) = -12$ and thus $-f(-\frac{1}{2}) - f(\frac{2}{3}) = 12$ (b). Finally, let $x = \frac{2}{3}$, then $f(\frac{2}{3}) + f(\frac{1}{1-\frac{2}{3}}) = 24 \times \frac{2}{3}$. This simplifies to $f(\frac{2}{3}) + f(3) = 16$ (c). Adding equations (a), (b) and (c) leads to $2 \times f(3) = 72 + 12 + 16$. Therefore $f(3) = 50$.

25. Three semicircles, each of area 24, overlap as shown in the diagram. The centres of the arcs are X, Y and Z and $\angle ZXY = 30°$.

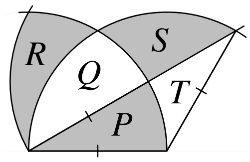
What is the total area of the shaded regions?

A 12 B $6\sqrt{3}$ C 15 D 18 E $8\sqrt{3}$

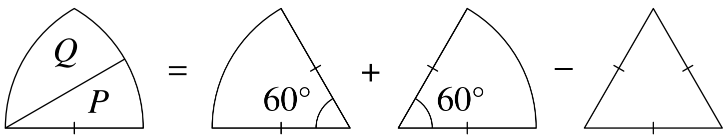


Solution **A**

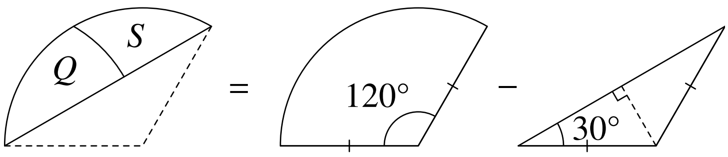
The diagram shows regions P, Q, R, S and T. As $\angle ZXY = 30°$, area of $P = \frac{1}{6}$ of the area of a semicircle $= \frac{1}{6} \times 24 = 4$. Regions $(P + Q)$ and $(Q + R)$ are congruent therefore area of $(P + Q)$ = area of $(Q + R)$ and so area of R = area of $P = 4$. We can also show that the area of $(P + Q)$ is the same as that of $(Q + S)$ as follows:



Region $(P+Q)$ can be deconstructed into two overlapping, congruent 60° sectors minus an equilateral triangle.



Region $(Q + S)$ can be deconstructed into a 120° sector, minus two 30°, 60°, 90° triangles whose total area is that of the equilateral triangle shown in the deconstruction of the region $(P + Q)$.



Therefore area of $(P + Q)$ = area of $(Q + S)$ and so area of P = area of $S = 4$. Hence the total area of the shaded region $= 4 + 4 + 4 = 12$.